ONE CASE OF THE EQUILIBRIUM OF A SYSTEM OF CRACKS IN AN ELASTO-BRITTLE MATERIAL

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In this article an approximate method of solving the problem of equilibrium of a system of parallel cracks in an elasto-brittle body is proposed.

Let an infinitely isotropic elastic body contain an infinitely large number of cracks parallel to the axis of abscissas and spaced at a distance of 2h between each other. Inside each crack a constant pressure p is acting along a segment a, but there is no stress acting on the remaining portion of the crack or at infinity. The crack length may be arbitrary. Due to the symmetry of the system our considerations may be confined to a band $0 \leq y \leq h$ whose lower edge coincides with the longitudinal crack axis, its upper edge being halfway between two adjacent cracks. The problem is to find the relations between p, h and a if all the elastic constants of the material and its specific surface energy are known.

Let us consider plane deformation. In this case the displacement vector components u and v, and the strain tensor components σ_x, σ_y, σ_{xy} are described in terms of analytical functions $\varphi(z)$, $\psi(z)$ by the Kolosov-Muskhelishvili formulas [1]:

$$2\mu (u + iv) = \varkappa\varphi(z) - \overline{z\varphi'(z)} - \psi(z) \qquad (\varkappa = 3-4\nu), \quad (1)$$

$$\sigma_x + \sigma_y = 4R\varphi'(z), \quad (2)$$

$$\sigma_y - \sigma_x + 2i\sigma_{xy} = 2[\bar{z}\varphi''(z) + \psi'(z)]. \quad (3)$$

Here ν is the Poisson ratio and μ is the shear modulus.

It should be noted that in view of the symmetry of the system

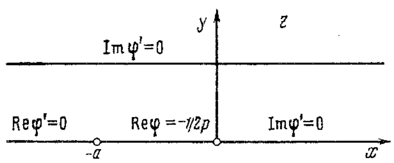

Fig. 1

and because of the preset boundary conditions the following condition is satisfied along the entire boundary of the region:

$$\sigma_{xy} = 0. \quad (4)$$

Moreover, along the boundary y = L and outside the crack along its continuation another condition

$$v = 0 \quad (5)$$

must be satisfied.

From Eq. (1) it follows that

$$2\mu \frac{\partial v}{\partial x} = (\varkappa + 1)\,\mathrm{Im}\,\varphi'(z) + \mathrm{Im}\,[\bar{z}\varphi''(z) + \psi'(z)], \quad (6)$$

and from Eq. (3)

$$\sigma_{xy} = \mathrm{Im}\,[\bar{z}\varphi''(z) + \psi'(z)]. \quad (7)$$

Consequently, if conditions (4) and (5) are satisfied at the same time along a segment of the boundary, as a result of Eqs. (6) and (7) at these segments we have

$$\mathrm{Im}\,\varphi'(z) = 0. \quad (8)$$

Furthermore, it is shown in problems of the equilibrium of a single crack [2] that along the crack and its continuation we have

$$\sigma_x = \sigma_y, \qquad y = 0. \quad (9)$$

In fact, one can introduce an analytical function $F(z) = z\varphi'' + \psi'$, whose limiting value at y = 0 coincides with the limiting value of the function in the right-hand part of Eq. (3). Then, by virtue of Eqs. (3) and (4), along the entire real axis Im F(z) = 0 and, consequently, $F(z) \equiv 0$ in the entire region. Hence, Eqs. (9) follow directly.

In the case of a system of cracks Eqs. (9) are not, generally speaking, satisfied. This follows clearly from the following physical considerations. If the cracks are sufficiently close to each other

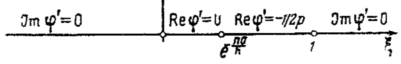

Fig. 2

($h/a \ll 1$), the material in the region adjacent to the loaded part of the crack will be in a state approaching uniaxial compression, for which

$$\sigma_x/\sigma_y = (1-\gamma)/\nu.$$

It may be postulated, however, that for cracks sufficiently distant from each other Eqs. (9) will be satisfied with a sufficient degree of accuracy. If this assumption is made, the following conditions should be satisfied:

$$R\varphi'(z) = -{}^1/_2 p, \quad (10)$$

for the loaded crack segments,

$$R\varphi'(z) = 0 \quad (11)$$

for the unloaded crack segments, and Eq. (8) for the remaining segments of the region boundary. Consequently, the problem is reduced to the well known Keldysh-Sedov problem of the theory of functions of complex variables [3].

As an example let us consider a problem analogous to that studied in [4]. Let a pressure p act on a segment $-a \leqslant x \leqslant 0$ of a semi-infinite crack situated on the negative part of the real axis. The region and the boundary conditions for the plane z are shown in Fig. 1. A function

$$\zeta = \xi + i\eta = e^{\pi z/h} \quad (12)$$

maps the band $0 \leq y \leq h$ on the upper semi-plane (Fig. 2). Now, we can obtain the solution with the aid of the Kedysh-Sedov formulas:

$$\varphi'(z) = \frac{p}{2\pi}\left(\frac{\zeta}{\zeta - 1}\right)^{1/2} \times$$

$$\times \left[2 \operatorname{arc\,tg} b - \left(\frac{1-\zeta}{\zeta}\right)^{1/2} \ln \frac{\sqrt{1-\zeta} + b\sqrt{\zeta}}{\sqrt{1-\zeta} - b\sqrt{\zeta}}\right]$$

$$\zeta = e^{\pi z/h}, \qquad b = \sqrt{e^{\pi a/h} - 1}. \quad (13)$$

In the crack tip $\varphi'(z)$ and, consequently, σ_y will have singularities in the form

$$\sigma_y = 2R\varphi'(z) = \frac{2p}{\pi}\left(\frac{h}{\pi x}\right)^{1/2} \operatorname{arc\,tg} b \equiv \frac{N_0}{\sqrt{x}}. \quad (14)$$

In accordance with [2], a crack will be in the state of equilibrium only if the following equations are satisfied:

$$N_0 \equiv \frac{K}{\pi}, \qquad K = \left(\frac{\pi E T}{1-\nu^2}\right)^{1/2}. \quad (15)$$

Here K is the cohesion modulus, E Young's modulus and T the specific surface energy.

From Eqs. (14) and (15) we obtain

$$\frac{p^2 h}{\pi\mu T}\,\mathrm{arc\,tg}^2\,b = \frac{\pi E T}{4(1-\nu^2)}$$

or, after elementary transformations,

$$\frac{\pi\mu T}{p^2 a} = \frac{h}{\pi a}\,\frac{c_1^2}{c_1^2 - c_2^2}\,\mathrm{arc\,tg}^2\,b$$

$$\left(c_1^2 = \frac{\lambda + 2\mu}{\rho}, \quad c_2^2 = \frac{\mu}{\rho}\right). \tag{16}$$

In [4], where a problem of steady-state propagation of a similar system of cracks was considered, the following expression was obtained for the boundary transition (in the static case) at $V \to 0$:

$$\frac{\pi\mu T}{p^2 a} = \frac{\mathrm{arc\,tg}\,b}{b} + \frac{h}{\pi a}\,\frac{c_2^2}{c_1^2 - c_2^2}\,\mathrm{arc\,tg}^2\,b\,. \tag{17}$$

Let us compare these two results. It should be pointed out first that an error (pointed out by A. M. Mikhailov) was made in [4]: the condition that tangential stresses at the unloaded crack segment must be equal to zero

$$\frac{\sigma_{xy}}{\mu} = \frac{2\beta_1 \alpha p}{\pi} \times$$

$$\times \left[\left(\frac{\xi_1}{1-\xi_1}\right)^{1/2} \mathrm{arc\,tg}\,b_1 - \left(\frac{\xi_2}{1-\xi_2}\right)^{1/2} \mathrm{arc\,tg}\,b_2\right],$$

$$\xi_i = \exp\frac{\pi x}{\beta_i h}, \qquad b_i = \left(\exp\frac{\pi a}{\beta_i h} - 1\right)^{1/2},$$

$$\alpha = \frac{1 + \beta_2^2}{\mu\left[(1+\beta_2^2)^2 - 4\beta_1\beta_2\right]},$$

$$\beta_i = \left(1 + \frac{V^2}{c_i^2}\right)^{1/2} \qquad (i = 1, 2), \qquad -a \leqslant x \leqslant 0 \tag{18}$$

was not rigorously satisfied.

Consequently, the solution found in [4] must be regarded as approximate. At $a/h \ll 1$, from Eqs. (18) we obtain accurate to the terms of the first order of smallness

$$\frac{\sigma_{xy}}{p} = \alpha\mu\left(\frac{a\,|x|}{h^2}\right)^{1/2}\left(1 - \frac{\beta_1}{\beta_2}\right).$$

Hence at $V \to 0$,

$$\frac{\sigma_{xy}}{p} = \frac{1}{2}\left(\frac{a\,|x|}{h^2}\right)^{1/2}, \qquad -a \leqslant x \leqslant 0\,. \tag{19}$$

At $a/h \to 0$ formulas (16) and (17) give the same result,

$$p^2 a/\pi\mu T = 1 - c_1^2/c_2^2\,.$$

This coincides with an expression obtained in [5] for a single crack. The relative difference in the crack length determined by formulas (16) and (17) at $a/h \ll 1$ is given (accurate to terms of the first order of smallness) by

$$\frac{\Delta a}{a} = \frac{\pi(c_1^2 - c_2^2)}{4c_1^2}\,\frac{a}{h}\,. \tag{20}$$

At $\lambda = \mu$ and $a/h = 5$ this value is approximately 10%, as is the maximum value of the ratio σ_{xy}/p determined by Eqs. (19).

The accuracy of the solutions of the static and dynamic problems in [4] has not yet been estimated.

REFERENCES

1. N. I. Muskhelishvili, Some Fundamental Problems of the Mathematical Theory of Elasticity [in Russian], 4th Edition, Izd-vo AN SSSR, 1954.

2. G. I. Barenblatt, "Mathematical theory of equilibrium cracks formed during brittle fracture," PMTF, no. 4, 1961.

3. M. A. Lavrent'ev and B. V. Shabat, Methods of the Theory of Functions of Complex Variables [in Russian], 2nd Edition, Fizmatgiz, 1958.

4. V. M. Kuznetsov, "Steady-state propagation of a system of cracks in an elasto-brittle material," PMTF, no. 3, 1964.

5. I. W. Craggs, "On the propagation of a crack in an elastic-brittle material," J. Mech. Phys. Solids, vol. 8, 1960.

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